# Computational Learning Theory
# FoPPS Logic and Learning School

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# What is Machine Learning?

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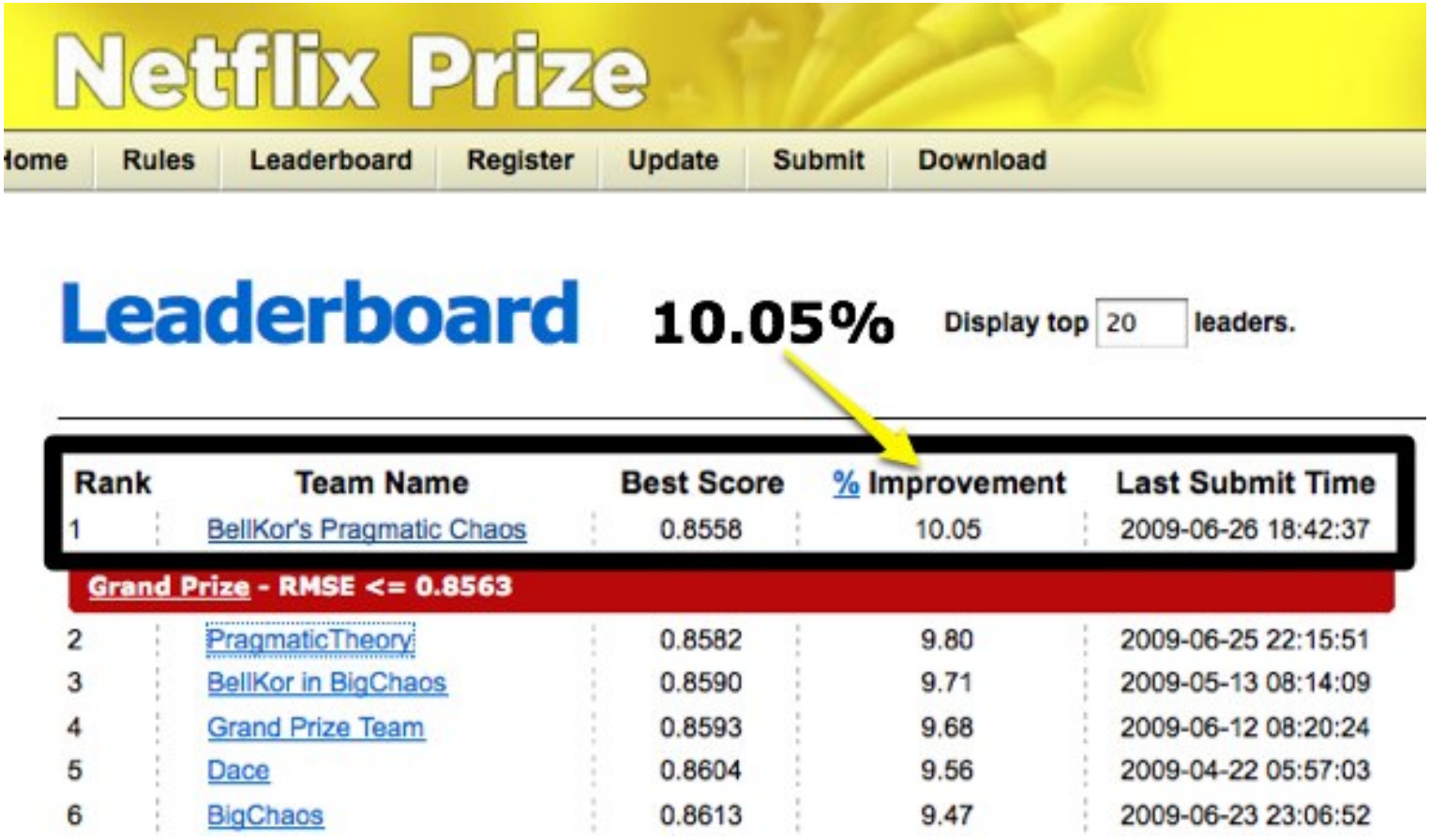
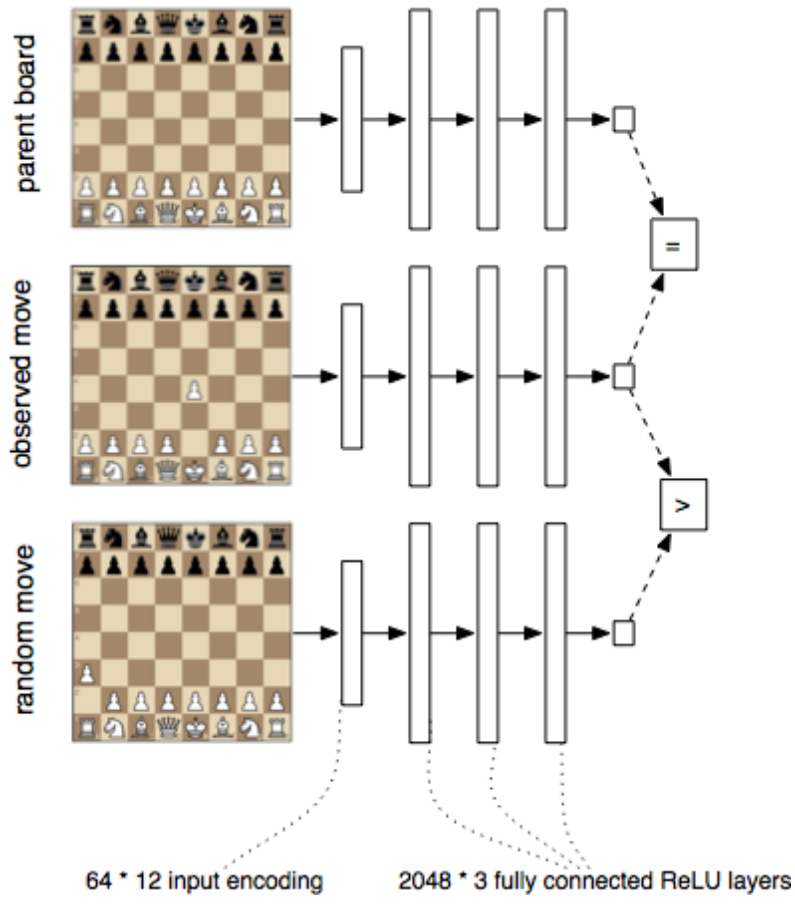
Have a computer solve problems by learning from data rather than being explicitly programmed.

# Filtering




Best algorithm for predicting user ratings, based on previous ratings, without any other information about the users or films.

# Planning and Strategy




Neural network trained to GM level using database of chess positions

# Machine Learning Overview

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- ***Ranking:*** order items according to several criteria, e.g., order the results of a search query by relevance to a particular user.

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**Applications to and connections with logic and verification!**

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# The PAC Model

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- Sample **training data** $S \in \mathcal{X}^m$ i.i.d. from $D$
- The desired output is a **hypothesis** $h \in \{0,1\}^{\mathcal{X}}$ such that

$$\mathrm{err}(h) \stackrel{\mathrm{def}}{=} \Pr_{x \sim D}\big(h(x) \neq c(x)\big).$$

  is "small".

# Example - Spam Filtering

```
Date:  Mon, 12 Oct 2015 19:36:45
From:  FSTTCS 2015
Subject:  Paper Submission 25
```
Not spam

```
Date:  13 Oct 2015 13:39:14
From:  HITACHI Personal Finance
Your Latest Money Saving Tips
```
Spam

```
Date:  Thu, 17 Sep 2015 05:00:12
From:  registration@it.ox.ac.uk
Subject:  Nexus mailbox expiry 01/10/15
```
???

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Represent data as set of **labelled feature vectors**:

| Bad Spelling | Attachment? | "Ben" | "Viagra" | **Spam?** |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | **1** |
| 0 | 1 | 1 | 0 | **1** |
| 0 | 1 | 0 | 1 | **0** |
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Input space is $\mathcal{X} = \{0,1\}^4$ and concept class $\mathcal{C} \subseteq \{0,1\}^{\mathcal{X}}$ is set of linear classifiers, e.g.,

$$2 \cdot \mathrm{BadSpell} - 3 \cdot \mathrm{Ben} + 1 \cdot \mathrm{Viagra} > 2 \ ?$$

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We say that $\mathcal{C}$ is **PAC learnable with sample complexity** $m$, **accuracy** $\varepsilon$, **and confidence** $\delta$ if there is a **learning map**

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s.t. for any target concept $c \in \mathcal{C}$ and distribution $D$ on $\mathcal{X}$,

$$\Pr_{S \sim D^m} \left( \mathrm{err}(H(S, c|_S)) \leq \varepsilon \right) \geq 1 - \delta \, .$$

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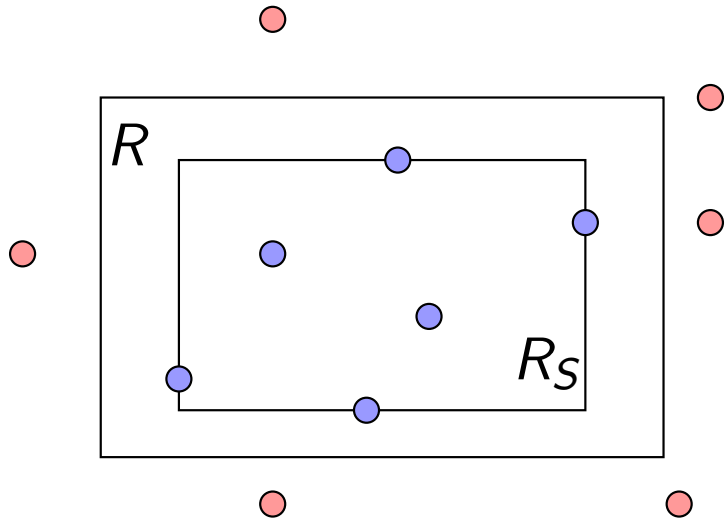
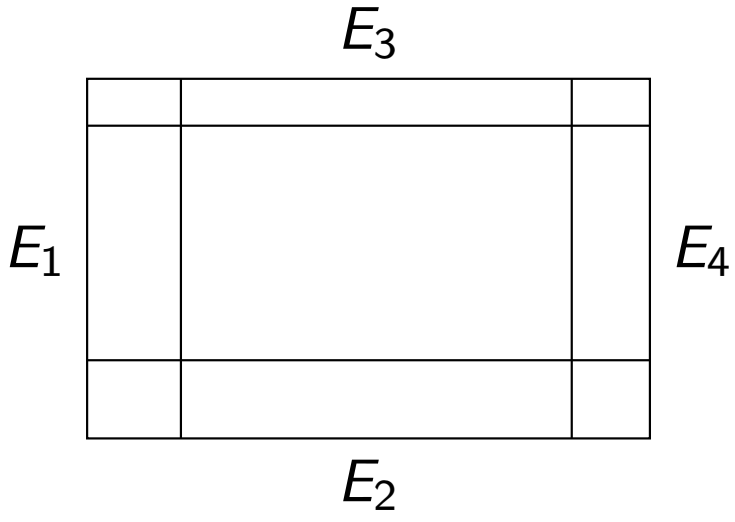
- PAC model presents learning as a problem of **prediction**.
- Doesn't explicitly reflect Occam's razor.

# Hypothesis Rectangle



# Border Regions




$$\Pr(E_1) = \Pr(E_2) = \Pr(E_3) = \Pr(E_4) = \varepsilon/4$$

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If the sample hits all four "border regions", then:

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**Learning map: output the "smallest" consistent hypothesis.**

# Building More Expressive Hypothesis Classes

# Perceptrons

A **linear classifier** is a function $f : \mathbb{R}^n \to \{-1, 1\}$ given by

$$f(\vec{x}) = \begin{cases} +1 & \text{if } \vec{a} \cdot \vec{x} \geq b \\ -1 & \text{otherwise} \end{cases}$$

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Consistency with $S = \{(\vec{x}_1, y_1), \ldots, (\vec{x}_k, y_k)\} \subseteq \mathbb{R}^n \times \{-1, 1\}$ formulated as a linear program:

$$\begin{aligned} \text{find} \quad & \vec{a} \in \mathbb{R}^n, b \in \mathbb{R} \text{ s.t.} \\ & y_i(\vec{a} \cdot \vec{x}_i - b) \geq 1, \; i = 1, \ldots, k. \end{aligned}$$

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If a consistent linear classifier exists then we say that $S$ is **linearly separable**.

# Combining Perceptrons

## Exercise

Given $S = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}^d$ and $F : S \to \{-1, +1\}$, how can we can realise $f$ as a composition of perceptrons?

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$$h(x) = \operatorname{sgn}\left(-1 + \sum_{i:F(x_i)=1} f_i(x) + g_i(x)\right)$$

# Layered Feedforward Neural Nets

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  for **weights** $w_i$ and **activation function** $\sigma$
- Activation functions:
  - Step $\sigma(z) = \mathbb{I}_{z \geq 0}$
  - Logistic sigmoidal $\sigma(z) = \frac{1}{1+e^{-z}}$
  - Rectified linear $\sigma(z) = \max(0, z)$

# But . . .

## Proposition

*The following problem is NP-hard. Given a feedforward neural network with $d$ inputs, a single hidden layer with 3 neurons, and step activation function, is there some weight setting that is consistent with a given labelled sample $S \subseteq \{0,1\}^d \times \{-1,+1\}$?*

# VC Dimension

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The *VC dimension* of $\mathcal{C}$ is defined by

$$\mathrm{VC}(\mathcal{C}) = \sup\left\{|S| : S \subseteq_{\mathrm{fin}} \mathcal{X} \text{ shattered by } \mathcal{C}\right\}.$$

# VC Dimension Workout

| $\mathcal{X}$ | $\mathcal{C}$ | $\mathrm{VC}(\mathcal{C})$ |
|---|---|---|
| $\{0,1\}^n$ | boolean formulas | |
| $\{0,1\}^n$ | boolean circuits with $n^3$ gates | |
| $\mathbb{R}^2$ | axis-aligned rectangles | |
| $\mathbb{R}^2$ | convex $k$-gons | |
| $\mathbb{R}^n$ | half spaces | |
| $\mathbb{R}$ | $\{x \mapsto \mathrm{sgn}(\sin \alpha x) : \alpha \in \mathbb{R}\}$ | |

# VC Dimension Workout

| $\mathcal{X}$ | $\mathcal{C}$ | $\mathrm{VC}(\mathcal{C})$ |
|---|---|---|
| $\{0,1\}^n$ | boolean formulas | $2^n$ |
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For "tame" models, VC dimension $\sim$ number of parameters

# Radon's Theorem

## Theorem (Radon's Theorem)

*Any set of $n+2$ points $S \subseteq \mathbb{R}^n$ can be partitioned into two subsets $S_1$ and $S_2$ such that the convex hulls of $S_1$ and $S_2$ intersect.*

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## Corollary

*No set of $n+2$ points in $\mathbb{R}^n$ can be shattered by halfspaces.*

# Concepts from Functions that are not Tame

Consider $S = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}$, where $x_i = 2^{-i}$ for $i = 1, \ldots, m$, and an arbitrary labelling $f : S \to \{-1, +1\}$.

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$$\alpha x_k \bmod 2\pi \quad = \quad \alpha 2^{-k} \bmod 2\pi$$

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$$\begin{aligned} \alpha x_k \bmod 2\pi &= \alpha 2^{-k} \bmod 2\pi \\ &= \frac{\pi}{2}\left(2^{-k} + \sum_{i=1}^{k-1}(1 - f(x_i))2^{i-k}\right) + \frac{\pi}{2}(1 - f(x_k)) \end{aligned}$$

# Concepts from Functions that are not Tame

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where $c \in (0, 1)$.

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\end{aligned}$$

where $c \in (0, 1)$. Thus $\operatorname{sgn}(\sin \alpha x_k) = f(x_k)$ for $k = 1, \ldots, m$.

# Dual Classes

Given $\mathcal{C} \subseteq \{0,1\}^{\mathcal{X}}$, the **dual class** $\mathcal{C}^* \subseteq \{0,1\}^{\mathcal{C}}$ consists of $f_x : \mathcal{C} \to \{0,1\}$, $x \in \mathcal{X}$, such that $f_x(c) = c(x)$ for all $c \in \mathcal{C}$.

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## Proposition

$\mathrm{VC}(\mathcal{C}) \leq 2^{\mathrm{VC}(\mathcal{C}^*)}$.

## Proof.

Suppose $\{c_1, \ldots, c_{2^n}\} \subseteq \mathcal{C}$ is shattered by $\mathcal{C}^*$ for some $n \in \mathbb{N}$. Then for each $i \in \{1, \ldots, n\}$ there exists $x_i \in \mathcal{X}$ such that $f_{x_i}(c_j) = 1$ if and only if the $i$-th bit of $j$ is 1. But then $\{c_1, \ldots, c_{2^n}\}$ shatters $\{x_1, \ldots, x_n\}$. $\square$

# The Growth Function

Consider a concept class $\mathcal{C}$ on input set $\mathcal{X}$. Given a finite sample $S \subseteq \mathcal{X}$, define

$$\Pi_{\mathcal{C}}(S) = \{c|_S : c \in \mathcal{C}\},$$

where $c|_S$ denotes the restriction of a function $c : \mathcal{X} \to \{0, 1\}$ to the set $S$.

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The *growth function* of $\mathcal{C}$ is defined as

$$\Pi_{\mathcal{C}}(m) = \max_{S:|S|=m} |\Pi_{\mathcal{C}}(S)|.$$

# Growth-Function Workout

Let $\mathcal{C}$ be the class of annular disks in $\mathbb{R}^2$ of the form $\{(x, y) : r \leq x^2 + y^2 \leq R\}$, where $r, R \in \mathbb{R}$. Describe the growth function $\Pi_{\mathcal{C}}(m)$.

# Sauer's Lemma

## Lemma

*Let $\mathcal{C}$ be a hypothesis set with finite VC dimension $d$. Then for all $m \geq d$,*

$$\Pi_{\mathcal{C}}(m) \leq \sum_{k=0}^{d} \binom{m}{k} \leq \left(\frac{em}{d}\right)^{d}.$$

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Considering $\mathcal{X} = \mathbb{N}$ and $\mathcal{C}$ the collection of subsets of $\mathbb{N}$ of cardinality at most $d$, one sees that the upper bound in Lemma 3 is tight.

# VC Dimension of Neural Nets with Step Activation

**Theorem**

*Fix an architecture with $n_0$ inputs and $\omega$ parameters and write $\mathcal{C} \subseteq \{0,1\}^{\mathbb{R}^{n_0}}$ the for class of functions that can be implemented by instantiating the parameters. Then*

$$\mathrm{VC}(\mathcal{C}) \leq 2\omega \log_2(e\omega)\,.$$

# Concept Classes from Predicate Logic

Given a $\sigma$-formula $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)$, $\sigma$-structure $\mathcal{A}$, and elements $b_1, \ldots, b_n \in A$, write

$$\varphi(\mathcal{A}, b_1, \ldots, b_n) := \{(a_1, \ldots, a_m) \in A^m : \mathcal{A} \models \varphi(a_1, \ldots, a_m, b_1, \ldots, b_n)\}$$

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Then the **definable family of sets**

$$\mathcal{C}(\varphi, \mathcal{A}) = \{\varphi(\mathcal{A}, b_1, \ldots, b_n) : b_1 \ldots, b_n \in A\}$$

specifies a concept class on the input space $\mathcal{X} = A^m$.

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Write $\mathrm{VC}(\varphi, \mathcal{A})$ for $\mathrm{VC}(\mathcal{C}(\varphi, \mathcal{A}))$.

# Semi-Algebraic Families (I)

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Key tool is:

## Theorem (Warren 68, Goldberg and Jerrum 95)

*Let $P_1, \ldots, P_\ell$ be polynomials of degree at most $d$ in $k$ real variables with $\ell \geq k$. Then the number of realisable sign assignments (each $P_i$ is positive, negative, or zero) is at most $(8ed\ell/k)^k$.*

# Semi-Algebraic Families (II)

## Proposition

*Let* $\mathcal{A} = (\mathbb{R}, 0, 1, +, \times)$ *and let* $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_k)$ *be a Boolean combination of* $s$ *polynomial equalities and inequalities, with each polynomial mentioned in* $\varphi$ *having degree at most* $d$*. Then* $\mathrm{VCD}(\varphi, \mathcal{A})$ *is at most* $2k \log(8eds)$*.*

# Useful Lemma

## Lemma (Shelah 71, Laskowski 92)

*Let $K$ be a class of structures such that for every formula $\varphi(x, y_1, \ldots, y_n)$ the set $\{\mathrm{VC}(\varphi, \mathcal{A}) : \mathcal{A} \in K\}$ is bounded. Then it also holds that $\{\mathrm{VC}(\varphi, \mathcal{A}) : \mathcal{A} \in K\}$ is bounded for every first-order formula $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)$.*

# o-Minimal Structures

## Proposition

*If $\mathcal{A}$ is an o-minimal structure then $\mathrm{VC}(\varphi, \mathcal{A}) < \infty$ for every first-order formula $\varphi$.*

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But by general results about o-minimal structures, there is a bound on the number intervals comprising $\varphi(\mathcal{A}, b_1, \ldots, b_n)$ that is independent of $b_1, \ldots, b_n \in A$. $\square$

# Neural Nets - Sigmoidal and Piecewise Polynomial Activation

**Theorem (Goldberg and Jerrum 95, Karpinski and Macintyre 97)**

*Consider a fixed architecture of layered feed-forward neural net with $N$ neurons and $\omega$ weights. The VC dimension of the corresponding class of functions is:*

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- $O(\omega^2 N^2)$ *with sigmoidal activation function.*

# Classes of Structures

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**Example.** Consider formula $\varphi := E(x, y)$ and $\mathcal{G}_n = (U, V, E)$ a bipartite graph, where $U = \{1, \ldots, n\}$ and $V = 2^{\{1,\ldots,n\}}$ and

$$E = \{(i, \alpha) : i \in U, \alpha \in V, i \in \alpha\}.$$

Then $\mathrm{VC}(\varphi, \mathcal{G}_n) \geq n$.

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□

# Bounds for MSO

## Theorem (Grohe and Turán 04)

*Let $\sigma$ be a relational signature and $K$ a class of structures of bounded tree-width. Then for any formula $\varphi$ of MSO, $\{\mathrm{VC}(\varphi, \mathcal{A}) : \mathcal{A} \in K\}$ is bounded.*

# Grid Graphs

Consider the $2^{\times} n$ grid $\mathcal{G}_{n,2^n}$. We describe a formula $\varphi(x, y)$ s.t. $\mathrm{VC}(\varphi, \mathcal{G}_{n,2^n}) \geq n$. Formula $\varphi$ is such that $\mathcal{G}_{n,2^n} \models \varphi((i, 1), (1, j))$ if the $i$-th bit in the binary expansion of $j$ is 1.

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Then $\mathcal{C}(\varphi, \mathcal{G}_{n,2^n})$ shatters the set $\{(1,1), \ldots, (n,1)\}$.

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## Theorem

*Let $\mathcal{C}$ be a concept class of VC dimension $d$ on input set $\mathcal{X}$. Let $\varepsilon, \delta > 0$ and $m \in \mathbb{N}$ be such that*

$$m \geq \frac{8}{\varepsilon}\left(d \log \frac{8}{\varepsilon} + \log \frac{2}{\delta}\right) \tag{1}$$

*Then for any target concept $c \in \mathcal{C}$ and distribution $D$ on $\mathcal{X}$, the probability that a sample of size $m$ is consistent with some hypothesis $h \in \mathcal{C}$ with $\mathrm{err}(h) \geq \varepsilon$ is at most $\delta$.*

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## Corollary

*VC classes are PAC learnable.*

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Two samples $S_1$ and $S_2$ of size $m$ drawn i.i.d. from $D$ behave similarly w.r.t. every statistical test in $\mathcal{C}$:

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- Set $S_1:=\{x_1,\ldots,x_m\}$ and $S_2:=\{x_{m+1},\ldots,x_{2m}\}$.
- What is the distribution on $S_1$ and $S_2$?

# Interlude - Chernoff Bound

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## Theorem

*Given* $0 \leq \gamma \leq 1$ *the following bounds hold:*

$$\begin{aligned} \Pr\left(S > (1+\gamma)p\right) &\leq e^{-mp\gamma^2/3} \\ \Pr\left(S < (1-\gamma)p\right) &\leq e^{-mp\gamma^2/2} \end{aligned}$$

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$$Pr(B) = \Pr(A \cap B) = \Pr(B \mid A) \Pr(A) \geq \frac{1}{2} \Pr(A)$$

But we know $\Pr(B) \leq \delta/2$, so $\Pr(A) \leq \delta$.

# Lower Bounds

**Theorem**

*Let $\mathcal{C}$ be a concept class that has VC dimension at least $d$. Then for any learning algorithm there exists a target concept $c \in \mathcal{D}$ and distribution $D$ on the input space $\mathcal{X}$ such that if the algorithm is given $d/2$ examples then the output hypothesis $h \in \mathcal{C}$ is such that*

$$\Pr(\mathrm{err}(h) > 1/8) > 1/8\,.$$

# Remember this Slide for Later!

**Theorem**

*Let $\mathcal{C} \subseteq \{0,1\}^{\mathcal{X}}$ have VC dimension $d$. Let $D$ be a distribution on $\mathcal{X}$ and $\varepsilon > 0$. Then there exists a multiset $S \subseteq \mathcal{X}$ of cardinality $O(d/\varepsilon^2)$ such that for all $c \in \mathcal{C}$,*

$$\left| \Pr_{x \sim D}(c(x) = 1) - \frac{1}{|S|} \sum_{x \in S} \mathbb{I}(c(x) = 1) \right| < \varepsilon \,.$$

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If $\mathcal{C}$ contains the zero function then a corollary of sample complexity bounds.

# Sample Compression Schemes

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mapping $(S, f)$ to $((S', f'), \sigma)$, where $S' \subseteq S$ and $f' = f|_{S'}$.

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- A *reconstruction map*

$$\rho : L_{\mathcal{C}}(k) \times I \to \{0,1\}^{\mathcal{X}},$$

  such that $\rho(\kappa(S, f))|_S = f$ for all $(S, f) \in L_{\mathcal{C}}(m)$, $m \geq k$.

# Sample Compression Scheme Workout

- Rectangles in the plane

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- Mistake-bounded mistake-driven online learning algorithms

# Excursion into Online Learning

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4. Goto 2.

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3. Total number of mistakes is at most $2 + 3r(1 + \log_2 n)$.

# Sample Compression Implies Learnability

## Theorem

*Let $\mathcal{C}$ be a concept class on instance space $\mathcal{X}$ that has a sample compression scheme $\kappa, \rho$ of kernel size $k$. Given $\varepsilon > 0$ and $\delta > 0$, let*

$$m \geq \max\left(\frac{2}{\varepsilon}\left(\log\left(\frac{2}{\delta}\right) + \log|I|\right), \frac{4k}{\varepsilon}\log\left(\frac{4k}{\varepsilon}\right) + 2k\right).$$

*Then the function $H : L_{\mathcal{C}}(m) \to \{0,1\}^{\mathcal{X}}$ defined by $H(S, f) = \rho(\kappa(S, f))$ is a PAC learning map with generalisation error $\varepsilon$ and failure probability $\delta$.*

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Reconstruction map receives only a small fraction of sample and so cannot overfit.

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**Theorem (Moran, Yehudayoff, 2016)**

*A concept class of VC dimension $d$ and dual VC dimension $d^*$ admits a sample compression scheme of kernel size $O(d \cdot d^*)$.*

# A "Caratheodory Theorem" for $\mathcal{C}$

## Proposition

*Let $\mathcal{C} \subseteq \{0,1\}^{\mathcal{X}}$ be a concept class such that the dual class $\mathcal{C}^* \subseteq \{0,1\}^{\mathcal{C}}$ has VC dimension $d^*$. Let $D^*$ be a distribution over $\mathcal{C}$ and $\varepsilon > 0$. Then there is a multiset $\mathcal{F} \subseteq \mathcal{C}$ of size $O(d^*/\varepsilon^2)$ such that for all $x \in \mathcal{X}$,*

$$\left| \Pr_{c \sim D^*}(c(x) = 1) - \frac{1}{|\mathcal{F}|} \sum_{c \in \mathcal{F}} \mathbb{I}\{c(x) = 1\} \right| \leq \varepsilon \,.$$

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What is the one-line proof?

# Zero-Sum Matrix Games

Rock-Paper-Scissors:

| | R | P | S |
|---|---|---|---|
| R | 0 | $+1$ | $-1$ |
| P | $-1$ | 0 | $+1$ |
| S | $+1$ | $-1$ | 0 |

# Minimax Theorem

## Theorem (Von Neumann Minimax Theorem)

*For any $m \times n$ matrix $M$,*

$$\min_{\vec{p} \in \Delta_m} \max_{\vec{q} \in \Delta_n} p^\top M_{i,j} q = \max_{\vec{q} \in \Delta_n} \min_{\vec{p} \in \Delta_m} p^\top M_{i,j} q \,.$$

# Main Result

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- There exists $s \in \mathbb{N}$ and a learning map $H : L_{\mathcal{C}}(s) \to \mathcal{C}$ s.t. for all $c \in C$ and every distribution $D$ on $\mathcal{X}$

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$$\Pr_{T \sim D^s} \operatorname{err}(H(T, c|_T)) \leq 1/3\,.$$

- Let $(S, c|_S)$ be a labelled sample to be compressed, where $S \subseteq \mathcal{X}$. Let $\mathcal{H} = \{H(T, c|_T) : T \subseteq S, |T| \leq s\}$ be the image of the learning map on subsets of $S$ of cardinality at most $s$.

# Central Claim

**Claim.** There are $k = O(d^*)$ functions $f_1 \ldots, f_k \in \mathcal{H}$ such that

$$|\{i : f_i(x) = c(x)\}| > k/2$$

for all $x \in S$.

# Compression and Reconstruction

**Compression.** Let $Z_i \subseteq S$ be such $f_i = H(Z_i, c|_{Z_i})$ for $i = 1, \ldots, k$. Define $\kappa(S, c|_S) = ((Z, c|_Z), \sigma)$ where $Z = \cup_{i=1}^{k} Z_i$ and $\sigma$ identifies which elements of $Z$ lie in each of the subsets $Z_1, \ldots, Z_k$.

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**Reconstruction.** To $h := \rho((Z, f), \sigma)$, consider the sets $Z_1, \ldots, Z_k$ determined by $Z$ and $\sigma$ and define $f_i = H(Z_i, c|_{Z_i})$ for $i = 1, \ldots, k$. For every $x \in \mathcal{X}$, we define $h(x)$ to be the majority element in the list $f_1(x), \ldots, f_k(x)$.

# Concepts that are Hard to Learn

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Is there a parameterized family of concept classes $\mathcal{C} = \{\mathcal{C}_n : n \in \mathbb{N}\}$, with $\mathrm{VC}(\mathcal{C}_n) = n^{O(1)}$, such that $\mathcal{C}$ admits no PAC learning algorithm that runs in polynomial-time in $n$ and the length of the representation of the target concept?

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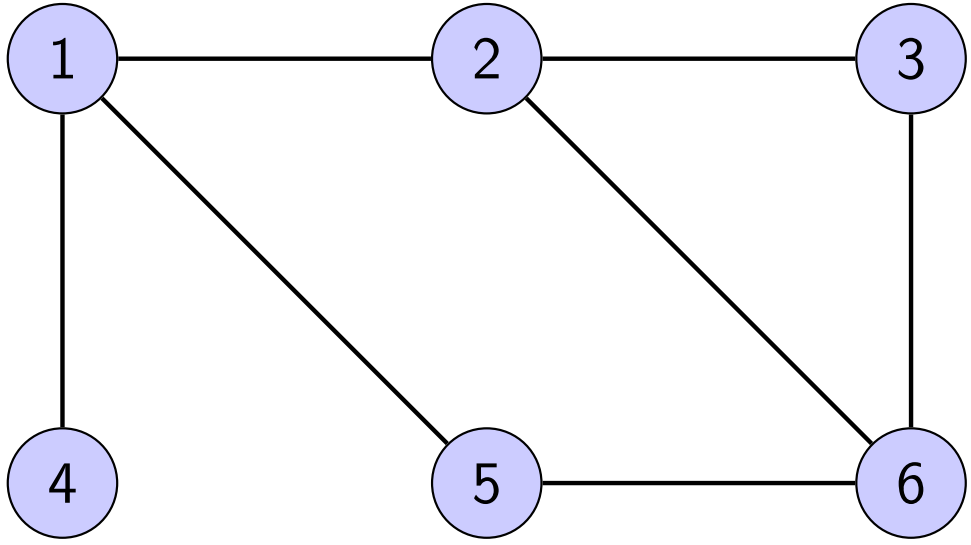
For example, what about $\mathcal{C}_n \subseteq \{0,1\}^{\Sigma^*}$ the class of regular languages whose minimal DFA has $n$ states?

# Consistency Problem for 3-Term DNF



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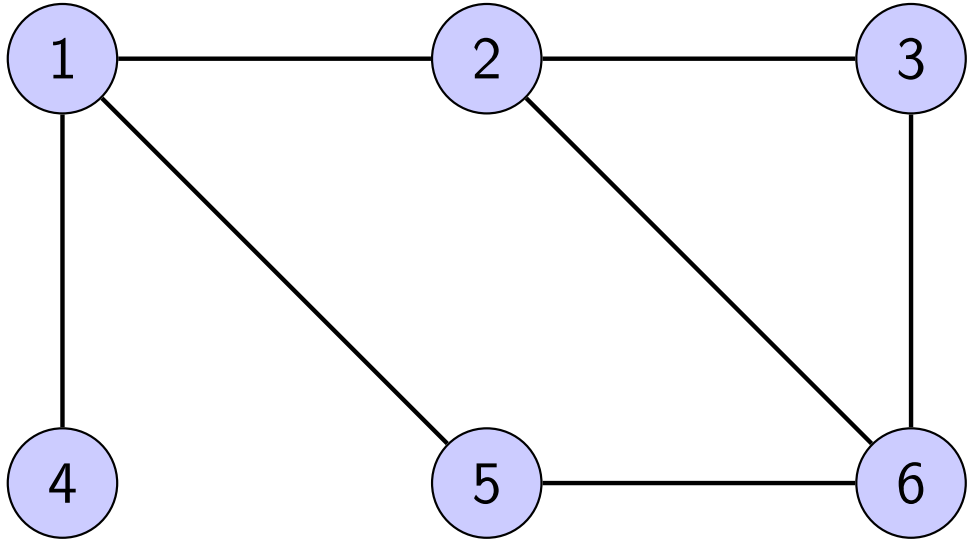


| Positive examples |
|---|
| $(100000, \mathbf{1})$ |
| $(010000, \mathbf{1})$ |
| $(001000, \mathbf{1})$ |
| $(000100, \mathbf{1})$ |
| $(000010, \mathbf{1})$ |
| $(000001, \mathbf{1})$ |

| Negative examples |
|---|
| $(110000, \mathbf{0})$ |
| $(100100, \mathbf{0})$ |
| $(100010, \mathbf{0})$ |
| $(011000, \mathbf{0})$ |
| $(010001, \mathbf{0})$ |
| $(001001, \mathbf{0})$ |
| $(000011, \mathbf{0})$ |

# Finding a Consistent Hypothesis

Suppose $G$ is 3-colourable then

$$\bigwedge_{i:i \text{ not red}} \neg P_i \ \vee \bigwedge_{i:i \text{ not blue}} \neg P_i \ \vee \bigwedge_{i:i \text{ not green}} \neg P_i$$

is a consistent hypothesis.

# Improper Learning via 3-Term DNF

- By distributivity, every 3-term DNF $\tau_1 \vee \tau_2 \vee \tau_3$ has an equivalent 3-CNF:

$$\tau_1 \vee \tau_2 \vee \tau_3 \equiv \bigwedge_{\ell_1 \in \tau_1, \ell_2 \in \tau_2, \ell_3 \in \tau_3} (\ell_1 \vee \ell_2 \vee \ell_3)\,.$$

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  - What is the running time?

# Some Elementary Number Theory

- Let $N = pq$ where primes $p, q \equiv 2 \bmod 3$. Then $\mathbb{Z}_N^*$ has order $\varphi(N) = (p-1)(q-1)$.

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- The inverse function is $f_N : \mathbb{Z}_N^* \to \mathbb{Z}_N^*$, where $g_N(y) = y^d \bmod N$ with $d = 3^{-1} \bmod \varphi(N)$.

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- The inverse function is $f_N : \mathbb{Z}_N^* \to \mathbb{Z}_N^*$, where $g_N(y) = y^d \bmod N$ with $d = 3^{-1} \bmod \varphi(N)$.
- Given $N$ and $x$, computing $g_N(x)$ is in polynomial time if we know both $p$ and $q$.

# Concepts that are Hard to Learn

## Definition (Discrete Cube Root Assumption)

For any polynomial $P(\cdot)$, there does not exist any algorithm, $A$, that runs in time $P(n)$ and on input $N$ and $x$, where $N$ is the product of two random $n$-bit primes $p, q \equiv 2 \bmod 3$ and $x$ is chosen randomly from $\mathbb{Z}_N^*$, outputs $g_N(x)$ with probability at least $1/P(n)$. The probability is over the random choices of $p$, $q$, $x$ and any internal randomisation of $A$.

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Discrete cube roots are hard to find on average.

# Discrete Cube Roots as a Learning Problem

Suppose that we have access to a sample, $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$, where $y_i = g_N(x_i)$ for $i = 1, \ldots, m$, and $x_i$ are drawn uniformly at random from $\mathbb{Z}_N^*$. Can we obtain $h : \mathbb{Z}_N^* \to \mathbb{Z}_N^*$, such that for $x$ drawn uniformly at random from $\mathbb{Z}_N^*$, it holds with probability at least $1 - \delta$ over the sample, that $\Pr(h(x) \neq g_N(x)) \leq \epsilon$?

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If our learning algorithm had polynomial sample complexity and running time (in the bit length of of $N$) then we would violate DCRA. Why???

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- Let $g_{N,i} : \mathbb{Z}_N^* \to \{0,1\}$ give the $i$-th bit of $g_N$ for $i = 1, \ldots, 2n$.

- A concept class that admits polynomial-size representations of $g_{N,i}$ and is evaluable in polynomial time is hard to learn under DCRA.

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**Theorem**

*There exists a fixed polynomial $P(\cdot)$, such that class of circuits, $\mathcal{C}$, where $\mathcal{C}_n$ consists of circuits of size at most $P(n)$ on $n$ inputs, is not PAC-learnable under the discrete cube root assumption.*

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What about DFA?

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Number of queries is polynomial in the size of the minimal automaton for the target and the logarithm of the longest counterexample given by the teacher.

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## Theorem (Carlyle and Paz, Fleiss)

*A function $f : \Sigma^* \to \mathbb{K}$ is recognizable by a weighted automaton if and only if its Hankel matrix has finite rank.*

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- A full-rank $n \times n$ submatrix of Hankel matrix of target function:

$$H = \begin{bmatrix} f(x_1y_1) & f(x_1y_2) & \cdots & f(x_1y_n) \\ f(x_2y_1) & f(x_2y_2) & \cdots & f(x_2y_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_ny_1) & f(x_ny_2) & \cdots & f(x_ny_n) \end{bmatrix}$$

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- $\boldsymbol{\eta} = \boldsymbol{e}_1$.

# Building a Hypothesis

The hypothesis is $(n, \Sigma, \boldsymbol{\alpha}, \{M(\sigma)\}_{\sigma\in\Sigma}, \boldsymbol{\eta})$, where

- $\boldsymbol{\alpha} = \boldsymbol{e}_1^\top H$.
- $\boldsymbol{\eta} = \boldsymbol{e}_1$.
- Each $M(\sigma)$ is defined by

$$HM(\sigma) = \begin{bmatrix} f(x_1\sigma y_1) & f(x_1\sigma y_2) & \cdots & f(x_1\sigma y_n) \\ f(x_2\sigma y_1) & f(x_2\sigma y_2) & \cdots & f(x_2\sigma y_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_n\sigma y_1) & f(x_n\sigma y_2) & \cdots & f(x_n\sigma y_n) \end{bmatrix}$$

# Parsing a Counterexample

## Proposition

*A counterexample $z$ has a prefix $w\sigma$, where $\sigma \in \Sigma$ and $w \in \Sigma^*$, such that for some $i \in \{1, \ldots, n\}$ the assignment $X \leftarrow X \cup \{w\}$, $Y \leftarrow Y \cup \{\sigma y_i\}$ increases the rank of $H$ by one.*

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## Theorem

*The concept class of weighted automata over a field $\mathbb{K}$ is exactly learnable with number of queries $O((|\Sigma| + \log m)r^2)$ where $r$ is the rank of the Hankel matrix of the target function and $m$ is the length of the longest counterexample returned by the teacher.*

# Summary

- Introduced a basic theoretical framework for supervised learning, emphasizing connections with logic and automata
- Further developments this week:
  - Statistical learning theory
  - Automata learning
  - Further connections with logic and verification
- Branching out:
  - Reinforcement learning
  - Bayesian learning